Research Statement

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July 15, 2023

1 Primary Project

My main topic of research in a broader sense is dynamical systems. I primarily focus on piecewise-smooth maps, funded under a Marsden grant 18-MAU-017 titled "Organised chaos: Using geometry to explain robust chaotic dynamics in switched dynamical systems" from Royal Society Te Apārangi.

Piecewise-linear maps can exhibit complicated dynamics yet are relatively amenable to an exact analysis. For this reason, they provide a useful tool for us to explore complex aspects of dynamical systems, such as chaos. They arise as approximations to specific types of grazing bifurcations of piecewise-smooth ODE systems [4], and are used as mathematical models, for example in social sciences [23]. In our work, we focus our studies on an abstract normal form, known as the *border-collision normal form* (a four-parameter family of continuous piecewise-linear maps on $\mathbb{R}^2$), given by

$$f_\xi(x,y) = \begin{cases} \big(\tau_L x + y + 1, -\delta_L x\big), & x \le 0, \\ \big(\tau_R x + y + 1, -\delta_R x\big), & x \ge 0, \end{cases} \tag{1}$$

This form includes (up to a change of variables) almost all continuous, piecewise-linear maps with two pieces, and does so with a minimal number of parameters, see [26]. It describes the dynamics created in *border-collision bifurcations* — where a fixed point of a piecewise-smooth map collides with a switching manifold [4, 25]. In this context, piecewise-linear maps have been used to explain bifurcations in diverse applications such as power converters [31] and mechanical systems with stick/slip friction [27]. The family (1) is a normal form in the sense that any non-degenerate, continuous, two-piece, piecewise-linear map on $\mathbb{R}^2$ can be converted to (1) via an affine change of coordinates [22].

The dynamics of the border-collision normal form can be very rich even in only two dimensions (i.e. when two scalar variables characterise the state space of the map), see [2, 15, 24]. Banerjee, Yorke, and Grebogi in their pioneering paper [3] identified an open parameter region $\Phi_{\mathrm{BYG}} \subset \mathbb{R}^4$ (to be defined below) throughout which the two-dimensional border-collision normal form has a chaotic attractor [17]. Region Φ_{BYG} is the region of *robust chaos*. Robust chaos refers to the phenomenon that a family of dynamical systems has a chaotic attractor throughout an open region of parameter space [30]. This does not occur for generic families of smooth one-dimensional maps as these have dense windows of periodicity [29], but is typical for systems with sufficiently many dimensions [18, 19], a well-known example being the Lorenz system [20, 28]. One can impose further requirements on the robustness, such as that the attractor varies continuously with respect to Hausdorff distance [16] or Lebesgue measure [1].

My project is organised as follows: In [13], we review the two-dimensional border-collision normal form and study the region Φ_{BYG}, where the map is invertible, orientation-preserving, and dissipative (for the most part) and apply renormalisation to partition this region by the number of connected components of a Chaotic Milnor attractor. This reveals previously undescribed bifurcation structure in a succinct way. Next, in [14] we further strengthen the existence of robust chaos throughout this region satisfying Devaney's definition of Chaos. We also show that the stable manifold of a saddle fixed point, despite being a one-dimensional object, densely fills an open region containing the attractor. Finally, we identify a heteroclinic bifurcation, not described previously, at which the attractor undergoes a crisis and may be destroyed. In [11] we construct a trapping region in phase space and an invariant expanding cone in tangent space, showing that the normal form exhibits a chaotic attractor throughout an open region of parameter space, this time generalising the construction to include the non-invertible and orientation-reversing normal form. The resulting parameter region is again open so the chaos is robust within this family of maps. The boundaries of the parameter region are where the construction fails in some way: three of these boundaries correspond to bifurcations where the chaotic attractor is destroyed, see Fig. 1. I am currently working on further extending the application of renormalisation to the non-invertible and orientation-reversing cases and on the identification of robust chaos in the $N-$dimensional border-collision normal form.

2 Secondary Projects

1. **Neurodynamics**: Recently my colleagues and I have been working on numerically analysing the dynamical properties of neuron models like Chialvo neurons [21, 12] and denatured Moris-Lecar neurons [6] under the influence of electromagnetic flux having various network topologies. In the future we plan to implement various control strategies to better understand how these neuron network models perform under various adaptive behaviour.

2. **Soccer-data modeling**: With my colleagues in University of Central Florida, I recently wrote a paper on developing a physics-driven kinemetic model for dominance space in soccer in terms of Voronoi diagrams [5]. We used the freely available Metrica Sports GPS data for this study. In the future we plan to extend our model to include pass intercepts

3. **Quantum Computation**: I am a physicist by training and during my masters degree I published a three-part expository article [8, 9, 10] on Quantum game theory where I reviewed and discussed the workflow of some famous quantum-game theoretic models. I maintain a corresponding R-package called *QGameTheory* [7], available in the CRAN repository. In terms of future plans, I intend to contribute towards the detection of chaotic dynamics in quantum mechanical systems, that could have potential applications in the field of quantum computation.

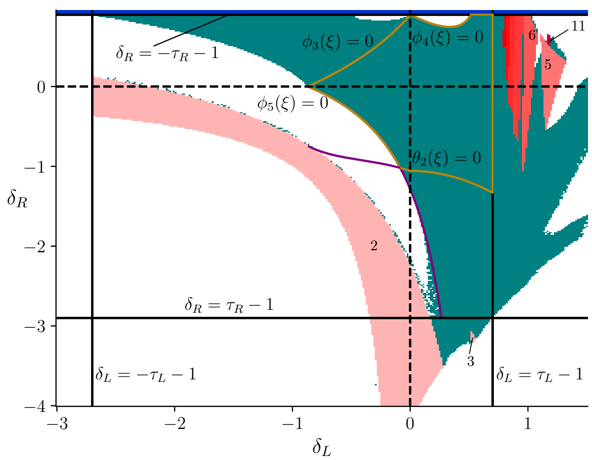

Figure 1: A two-parameter bifurcation diagram of (1) with $\tau_L = 1.7$ and $\tau_R = -1.9$. The boundary of $\Phi_{\mathrm{trap}} \cap \Phi_{\mathrm{cone}}$ is shown in yellow. The coloured regions show the result of a numerical simulation (white: no attractor; green: chaotic attractor; other colours: periodic attractor). Some periods are indicated (e.g. in the large pink region the map has a stable period-2 solution. In the blue strip at the top the fixed point X is stable.

References

[1] J. Alves, A. Pumariño, and E. Vigil. Statistical stability for multidimensional piecewise expanding maps. *Proc. Amer. Math. Soc.*, 145(7):3057–3068, 2017.

[2] S. Banerjee and C. Grebogi. Border collision bifurcations in two-dimensional piecewise smooth maps. *Phys. Rev. E*, 59(4):4052–4061, 1999.

[3] S. Banerjee, J.A. Yorke, and C. Grebogi. Robust chaos. *Phys. Rev. Lett.*, 80(14):3049–3052, 1998.

[4] M. di Bernardo, C. Budd, A.R. Champneys, and P. Kowalczyk. *Piecewise-smooth dynamical systems: theory and applications*, volume 163. Springer-Verlag, New York, 2008.

[5] C. J. Efthimiou, G. DeCamillis, and I. Ghosh. A physics-driven study of dominance space in soccer. *arXiv preprint arXiv:2202.00414*, 2022.

[6] H. O. Fatoyinbo, S. S. Muni, I. Ghosh, I. O. Sarumi, and A. Abidemi. Numerical bifurcation analysis of improved denatured morris-lecar neuron model. In *2022 international conference on decision aid sciences and applications (DASA)*, pages 55–60. IEEE, 2022.

[7] I. Ghosh. CRAN - package QGameTheory. `https://cran.r-project.org/web/packages/QGameTheory/index.html`, 2020.

[8] I. Ghosh. Quantum game theory—i: A comprehensive study. *Resonance*, 26(5):671–684, 2021.

[9] I. Ghosh. Quantum game theory—ii: A comprehensive study. *Resonance: Journal of Science Education*, 26(6), 2021.

[10] I. Ghosh. Quantum game theory—iii: A comprehensive study. *Resonance*, 26(7):939–951, 2021.

[11] I. Ghosh, R. I. McLachlan, and D. J. W. Simpson. Robust chaos in orientation-reversing and non-invertible two-dimensional piecewise-linear maps. *arXiv preprint arXiv:2307.05144*, 2023.

[12] I. Ghosh, S. S. Muni, and H. O. Fatoyinbo. On the analysis of a heterogeneous coupled network of memristive chialvo neurons. 2023.

[13] I. Ghosh and D. J. W. Simpson. Renormalization of the two-dimensional border-collision normal form. *Int. J. Bifurcation Chaos*, 32(12):2250181, 2022.

[14] I. Ghosh and D. J. W. Simpson. Robust devaney chaos in the two-dimensional border-collision normal form. *Chaos*, 32(4), 2022.

[15] P. Glendinning. Bifurcation from stable fixed point to 2d attractor in the border collision normal form. *IMA J. Appl. Math.*, 81(4):699–710, 2016.

[16] P. A. Glendinning and D. J. W. Simpson. Robust chaos and the continuity of attractors. *Trans. Math. Appl.*, 4(1):tnaa002, 2020.

[17] P. A Glendinning and D.J.W. Simpson. A constructive approach to robust chaos using invariant manifolds and expanding cones. *Discrete Contin. Dyn. Syst.*, 41(7):3367–3387, 2021.

[18] A. S. Gonchenko, S. V. Gonchenko, A. O. Kazakov, and A. D. Kozlov. Elements of contemporary theory of dynamical chaos: A tutorial. Part I. Pseudohyperbolic attractors. *Int. J. Bifurcation Chaos*, 28(11):1830036, 2018.

[19] Sergey Gonchenko, Alexey Kazakov, and Dmitry Turaev. Wild pseudohyperbolic attractor in a four-dimensional Lorenz system. *Nonlinearity*, 34(4):2018–2047, 2021.

[20] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Publ. Math. IHES*, 50:59–72, 1979.

[21] S. S. Muni, H. O. Fatoyinbo, and I. Ghosh. Dynamical effects of electromagnetic flux on chialvo neuron map: nodal and network behaviors. *Int. J. Bifurcation Chaos*, 32(09):2230020, 2022.

[22] H.E. Nusse and J.A. Yorke. Border-collision bifurcations including "period two to period three" for piecewise smooth systems. *Phys. D: Nonlinear Phenomena*, 57(1-2):39–57, 1992.

[23] T. Puu and I. Sushko. *Business Cycle Dynamics: Models and Tools.* Springer-Verlag, New York, 2006.

[24] D.J.W. Simpson. Sequences of periodic solutions and infinitely many coexisting attractors in the border-collision normal form. *Int. J. Bifurcation Chaos*, 24(06):1430018, 2014.

[25] D.J.W Simpson. Border-collision bifurcations in rˆn. *SIAM Rev.*, 58(2):177–226, 2016.

[26] D.J.W. Simpson. Detecting invariant expanding cones for generating word sets to identify chaos in piecewise-linear maps. *J. Difference Eq. Appl.*, pages 1–33, 2022.

[27] R. Szalai and H. M. Osinga. Invariant polygons in systems with grazing-sliding. *Chaos*, 18(2):023121, 2008.

[28] W. Tucker. The Lorenz attractor exists. *C. R. Acad. Sci. Paris*, 328(12):1197–1202, 1999.

[29] S. van Strien. One-parameter families of smooth interval maps: Density of hyperbolicity and robust chaos. *Proc. Amer. Math. Soc.*, 138(12):4443–4446, 2010.

[30] E. Zeraoulia and J. Sprott. *Robust chaos and its applications*, volume 79. World Scientific, 2012.

[31] Z. T. Zhusubaliyev and E. Mosekilde. Equilibrium-torus bifurcation in nonsmooth systems. *Phys. D*, 237(7):930–936, 2008.